Exam Calculus 1

3 February 2010, 9.00-12.00.

This exam has 6 problems. The maximum score per problem can be found below. Write on each page your name and student number, and on the first page your seminar group. The use of annotations, books and calculators is not permitted in this examination. All answers must be supported by arguments and/or work. Success.

1. (a) Formulate the principle of mathematical induction.

   (b) Prove that if $n \geq 1$ is a positive integer, then

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

2. Find all (complex) solutions of

   (a)

$$z^2 - iz + 1 = 0$$

   (b)

$$z^3 + 1 = 0$$

3. (a) Give the following definition:
   The derivative of a function f at a point a is defined by

$$f'(a) \stackrel{\text{def}}{=} \cdots$$

   (b) We assume that $f(x)$ is differentiable on $-\infty < x < \infty$.
   Give the following definition:
   The derivative f' is continuous at a point a if ...

   (c) A function f is called continuously differentiable on $(-\infty, \infty)$, if $f(x)$ is differentiable for all $x \in (-\infty, \infty)$ and the derivative $f'(x)$ is continuous for all $x \in (-\infty, \infty)$.
   Assume that f and g are continuously differentiable on $(-\infty, \infty)$ and $g'(a) \neq 0$, $f(a) = g(a) = 0$. Show that (l'Hospitals rule is correct):

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

   (d) Classical mechanics states that the displacement of a particle in a gravitational field is equal to $\frac{1}{2}gt^2$, where t denotes time. According to the theory of relativity the displacement caused by a constant g-force is given by

$$x(t) = \frac{c^2}{g}\left(\left[1 + \left(\frac{gt}{c}\right)^2\right]^{1/2} - 1\right).$$

Here c is the speed of light. Notice: c and g are constant.
Evaluate

$$\lim_{t \to 0} \frac{x(t)}{t^2}.$$

4. Show, using the ϵ-δ-definition of limit, that

$$\lim_{x \to 2} \frac{6}{x} = 3.$$

5. (a) Evaluate

$$\int_0^{\pi} \mathrm{e}^{2x} \sin(x)\, dx$$

   (b) Evalute

$$\int_0^4 \sqrt{x^4 + 9x^2}\, dx$$

6. Find all solutions $y(x)$ of the differential equation

   (a)

$$xy' - (x+1)y = 0$$

   (b) Idem,

$$xy' - (x+1)y = x$$

Maximum score:

1a	0.5	2a	0.5	3a	0.5	4	1.0	5a	0.7	6a	0.7
b	1.0	b	1.0	b	0.3			b	0.8	b	0.8
				c	0.6						
				d	0.6						